Math 5C Discussion Problems 2

Path Independence

1. Let C be the striaght-line path in $\mathbb{R}^2$ from the origin to $(3,1)$. Define $f(x,y) = xye^{xy}$.
   (a) Evaluate $\int_C \nabla f \cdot d\mathbf{r}$.
   (b) Evaluate $\int_C ((1,0) + \nabla f) \cdot d\mathbf{r}$.
   (c) Evaluate $\int_C ((y,0) + \nabla f) \cdot d\mathbf{r}$.
2. Let C be the positively oriented unit circle in $\mathbb{R}^2$ centered at the origin.
   (a) Evaluate $\oint_C y\,dx - x\,dy$.
   (b) Use the previous part to explain why $(y,-x)$ isn't the gradient of a function.
3. For each of the following vector fields, determine whether it is the gradient of a function.
   (a) $(4x^2 - 4y^2 + x, 7xy + \ln y)$ on $\mathbb{R}^2$
   (b) $(3x^2 \ln x + x^2, x^3/y)$ on $\mathbb{R}^2$
   (c) $(x^3 y, 0, z^2)$ on $\mathbb{R}^3$
4. For each of the following, find the function f.
   (a) $f(0,0,0) = 0$ and $\nabla f = (x,y,z)$
   (b) $f(1,2,3) = 4$ and $\nabla f = (5,6,7)$
   (c) $f(1,1,1) = 1$ and $\nabla f = (2xyz + \sin x, x^2 z, x^2 y)$
5. For which a is $(ax \ln y, 2y + x^2/y)$ the gradient of a function in (some subset of) $\mathbb{R}^2$?
6. Let C be a curve in $\mathbb{R}^2$ given by $\mathbf{r}(t) = (\cos^5 t, \sin^3 t, t^4)$, where $0 \le t \le \pi$. Evaluate $\int_C (yz, xz, xy) \cdot d\mathbf{r}$.

Green's Theorem

1. Let D be the unit disk centered at the origin in $\mathbb{R}^2$.

   (a) Evaluate $\displaystyle\oint_{\partial D} dx + x\,dy$.

   (b) Evaluate $\displaystyle\oint_{\partial D} \arctan(e^{\sin x})\,dx + y\,dy$.

   (c) Evaluate $\displaystyle\oint_{\partial D} (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$.

   (d) Evaluate $\displaystyle\oint_{\partial D} \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy$, given that $f_{xx} + f_{yy} = 0$.

2. Find the area of the following regions in $\mathbb{R}^2$ using Green's theorem.

   (a) The unit disk

   (b) The inverted cycloid: the region bounded by the x axis and the parametric curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi$

   (c) The astroid: $x^{2/3} + y^{2/3} \leq a^{2/3}$

   (d) The ellipse: $x^2/a^2 + y^2/b^2 \leq 1$

3. Let R be a region in the plane with ∂R positively oriented. On ∂R, $d\mathbf{r} = (dx, dy)$. Derive an expression for $\mathbf{n}\,ds$ in terms of dx and dy. Deduce (using the 'standard' Green's theorem) the normal form of Green's theorem:

$$\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\,ds = \iint_R \nabla \cdot \mathbf{F}\,dA.$$

   Notice that this is just a 2D version of the divergence theorem.

4. Let f be a smooth function and D be a disk in $\mathbb{R}^2$ with outward unit normal $\mathbf{n}$. For points on ∂D, denote $\partial f/\partial n$ to mean the directional derivative of f in the direction of $\mathbf{n}$. Prove that

$$\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \iint_D \Delta f\,dA.$$

5. Prove the identity $\displaystyle\oint_{\partial D} f\nabla f \cdot \mathbf{n}\,ds = \iint_D (f\Delta f + \|\nabla f\|^2)\,dA$.

6. Prove the identity $\displaystyle\oint_{\partial D} PQ\,dx + PQ\,dy = \iint_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dA$.

Divergence Theorem

1. In each of the following situations, evaluate $\iint_{\partial R} \mathbf{F} \cdot d\mathbf{A}$. Assume ∂R is outward oriented.

   (a) Let R be the unit ball centered at the origin and $\mathbf{F} = (x, 2y, 3z)$.

   (b) Let R be the unit cube $0 \leq x, y, z \leq 1$ and $\mathbf{F} = (y^2 + \sin z, e^{\sin z} + 2, xy + z)$.

   (c) Let R be the hemisphere $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ and $\mathbf{F} = (xz, yz, z^2)$.

2. Let S_1 be the disk $x^2 + y^2 \leq 1$, $z = 1$, oriented upward. Let S_2 be the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$, oriented downward. Together, S_1 and S_2 enclose a region R. Define $\mathbf{F} = (x + e^y, y + \cos x, z)$.

   (a) Find the flux of $\mathbf{F}$ across S_1 directly.

   (b) Integrate $\nabla \cdot \mathbf{F}$ over R.

   (c) With no extra computation, find the flux of $\mathbf{F}$ across S_2. Do you see how this problem could be generalized?

3. In each of the following, use the method of the previous problem to find the flux of $\mathbf{F}$ across S.

   (a) Let S be the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, outwardly oriented. Assume $\mathbf{F} = (x^2, 0, 2z)$.

   (b) Let S be the cone $z = 4 - \sqrt{x^2 + y^2}$, $z \geq 0$, oriented upward . Assume $\mathbf{F} = (xy, yz, xz)$.

4. Let R be a solid region with smooth boundary ∂R oriented outward. Assuming all functions are smooth, prove the following identities.

   (a) $\iint_{\partial R} \nabla \times \mathbf{F} \cdot d\mathbf{A} = 0$

   (b) $\iint_{\partial R} f \nabla g \cdot d\mathbf{A} = \iiint_R (f \Delta g + \nabla f \cdot \nabla g)\, dV$.

   (c) $\iint_{\partial R} (f \nabla g - g \nabla f) \cdot d\mathbf{A} = \iiint_R (f \Delta g - g \Delta f)\, dV$.

   (d) $\iint_{\partial R} (x, y, z) \cdot d\mathbf{A} = 3 \text{ volume}(R)$

5. Standard integration by parts is proved using the product rule. But there are many different product rules; each gives a different integration by parts.

   (a) Let D be a ball in $\mathbb{R}^3$ with outward unit normal vector $\mathbf{n}$. Assuming $\nabla \cdot (f\mathbf{G}) = f(\nabla \cdot \mathbf{G}) + \mathbf{G} \cdot \nabla f$, prove that

   $$\iiint_D f(\nabla \cdot \mathbf{G})\, dV = \iint_{\partial D} f\mathbf{G} \cdot d\mathbf{A} - \iiint_D (\nabla f) \cdot \mathbf{G}\, dV.$$

   (b) Now let D be the unit ball centered at the origin. Evaluate

   $$\iiint_D e^{-\sqrt{x^2+y^2+z^2}}\, \nabla \cdot \left(\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \right) dV.$$

   You can ignore the singularities at the origin (this could be made rigorous).

Stokes' Theorem

1. In each of the following situations, evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

   (a) Let S be the upper half ($z \geq 0$) of the sphere $x^2+y^2+z^2 = 1$, oriented upward, and $\mathbf{F} = (x, xz, ye^{\cos y})$.

   (b) Let S be the right half ($x \geq 0$) of the sphere $x^2+y^2+z^2 = 1$, oriented rightward, and $\mathbf{F} = (x^3, -y^3, 0)$.

   (c) Let S be the part of the plane $z = x$ with $x^2+2x+y^2 \leq 3$, oriented upward, and $\mathbf{F} = ((x+1)^2, 0, -x^2)$.

2. Let C be the interesection of a (nonvertical) plane and the cylinder $x^2 + y^2 = 4$ in $\mathbb{R}^3$. Show that

$$\oint_C (2x - y)\, dx + (2y + x)\, dy = 8\pi.$$

3. Let C be a simple, closed, smooth curve on the sphere $x^2+y^2+z^2 = 1$. Show that $\oint_C (-2xz, 0, y^2) \cdot d\mathbf{r} = 0$.

4. Let S be a smooth oriented surface with smooth boundary ∂S. Assuming all functions are smooth, prove the following identities.

   (a) $\oint_{\partial S} f\nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{A}$

   (b) $\oint_{\partial S} (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

5. Standard integration by parts is proved using the product rule. But there are many different product rules; each gives a different integration by parts.

   (a) Let S be a smooth oriented surface with boundary ∂S. Given a smooth vector field $\mathbf{G}$ and a smooth scalar function f, show that

$$\iint_S f(\nabla \times \mathbf{G}) \cdot d\mathbf{A} = -\iint_S (\nabla f \times \mathbf{G}) \cdot d\mathbf{A} + \oint_{\partial S} f\mathbf{G} \cdot d\mathbf{r}.$$

   (b) Now let S be the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, oriented downward. Define $\mathbf{G} = (-y, x, \arctan(xyz)e^{x^2})$ and evaluate

$$\iint_S z^2\, (\nabla \times \mathbf{G}) \cdot d\mathbf{A}$$

   (c) (Harder) Recall the identity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$. Prove the vector equation

$$\iint_S \nabla f \times d\mathbf{A} = -\oint_{\partial S} f d\mathbf{r}.$$

Sequences and Series

1. Find the limit of the following sequences.

   (a) $a_n = \ln n / n$

   (b) $a_n = (1 - 2/n)^{3n}$

   (c) $a_n = \sqrt{n^2 + 3n} - n$

2. Given the sequence (a_n):

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots$$

   show that $a_{n+1}^2 - 1 = a_n$. Given that the limit of the sequence exists, use this formula to find it.

3. Evaluate the following series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}}$

   (b) $1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \cdots$, where $0 < \theta < \pi/2$

   (c) $\displaystyle\frac{14}{15} + \frac{28}{75} + \frac{56}{375} + \frac{112}{1875} + \cdots$

4. Evaluate the following series.

   (a) $\displaystyle\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n(n+2)}{(n+1)^2}\right)$

   (d) $\displaystyle\sum_{n=0}^{\infty} \arctan(n+1) - \arctan(n)$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$

5. A difficult series to evaluate is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this fact to evaluate the following.

   (a) $\displaystyle\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

   (b) $\displaystyle 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Convergence of Series

1. Determine the convergence of the following series.

   (a) $\sum \frac{\sqrt{n}}{n^2}$

   (b) $\sum \frac{\sqrt{n}}{1+n^2}$

   (c) $\sum \left(1+\frac{1}{n}\right)^{-n}$

   (d) $\sum \left(1-\frac{1}{n}\right)^{n^2}$

   (e) $\sum \sin(1/n)$

   (f) $\sum n^2 e^{-n^3}$

   (g) $\sum \frac{\arctan n}{n^2}$

   (h) $\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$

   (i) $\sum \frac{k^{-1/2}}{1+\sqrt{k}}$

   (j) $\sum \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$

2. Determine whether each series converges absolutely, conditionally, or not at all.

   (a) $\sum \frac{(-1)^n}{n \ln n}$

   (b) $\sum \frac{(-4)^n}{n^2}$

   (c) $\sum \frac{\cos(n\pi)}{n}$

3. Use the error bound in the alternating series test to approximate, within 2 decimal place accuracy, the following values. The exact value of each series is given only as trivia.

   (a) $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$

   (b) $\cos(1/2) = 1 - \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!} - \frac{(1/2)^6}{6!} + \cdots$

4. Suppose we want to approximate $\ln 2$ with 2 digits of accuracy. If we use the alternating series test,

   (a) How many terms of $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ are needed?

   (b) How many terms of $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n 2^n}$ are needed?

5. For which real numbers p does the series

$$\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

   converge absolutely? Conditionally? Not at all?

Series: Miscellaneous

1. Define $S = 1 + 2/3 + 3/3^2 + 4/3^3 + 5/3^4 + \cdots$.
   (a) Show that the series converges.
   (b) Write out a series for $3S$.
   (c) Substract the given equation from the one you just wrote.
   (d) Evaluate S using the previous part.
2. There is a constant γ, called the Euler-Mascheroni constant, so that

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \epsilon_n,$$

   where $\epsilon_n \to 0$ when $n \to \infty$. Use this fact to answer the following.
   (a) Use the above formula to show that $\sum(1/n)$ diverges.
   (b) Evaluate $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$.
3. Find constants A and B so that

$$\frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{A2^k}{3^k - 2^k} + \frac{B2^k}{3^{k+1} - 2^{k+1}}.$$

   Use this to evaluate

$$\sum_{n=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}.$$

4. The Cauchy Condensation test states that, given a positive nonincreasing sequence a_n, $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. Use this test to check convergence of the following:
   (a) $\sum 1/n$
   (b) $\sum 1/(n \log_2 n)$
   (c) $\sum 1/(n(\log_2 n)(\log_2 \log_2 n))$
5. The Fibonacci numbers form a sequence F_n, where $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all integers n.
   (a) Use telescoping to evaluate $\sum_{n=1}^{\infty} \frac{F_{n-1}}{F_n F_{n+1}}$.
   (b) It turns out that (amazingly)

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

   Does $\sum F_n^{-1}$ converge?
6. In 1914, Ramanujan proved that

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

   Show that this series converges.